Optimal growth under a climate constraint

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January 2013

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‡The research leading to these results has received funding from European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement n 266992 - Global-IQ "Impacts Quantification of global changes".

§Paper presented at the 20th annual EAERE 2013 Toulouse Conference, 26-29th June 2013.

Abstract

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Inside a standard growth model with exhaustible resources, we study the optimal growth policy of an economy submitted to a climate constraint, taking the form of a ceiling over admissible atmospheric carbon concentrations. The optimal scenario is a three phases path: a rise of carbon concentrations until the carbon cap is attained followed by a time phase constrained by the ceiling on possible emissions and a last unconstrained phase of resource depletion. Depending upon the primitives of the model we show that the optimal path may be of two main kinds: paths characterized by a positive growth of the economy and paths corresponding to a complex structural adjustment process involving negative growth during some time interval.

Keywords: Carbon pollution; economic growth; exhaustible resources.
JEL classifications: Q00, Q32, Q43, Q54.

Contents

1 Introduction

The dooming predictions of the Meadows (1972) report (the so-called 'Club de Rome' report) raised strong reactions inside the economist's profession, culminating in the Review of Economic Studies Symposium in 1974. The important contributions appearing in this special issue from Stiglitz, Dasgupta and Heal, Solow, among others, largely formed the basis of contemporary exhaustible resources economics until today. The economists attitude after the publication of the first alarming reports of the IPCC concerning the future climate of the planet has been quite different. Most efforts have been devoted to incorporate climate dynamics inside more or less sophisticated optimal growth frameworks, an approach pioneered by Nordhaus forty years ago. This has been the starting point of the development of integrated assessment models (IAM). These models mimic the climatologists approach of running numerical simulations of the future climate and economic conditions throughout the world for the current century and beyond. However, and this is a striking difference with the exhaustibility debate of the seventies, a considerable less effort has been devoted to analyze in conceptual terms the challenge of climate change for future growth.

One reason is the analytical complexity of the problem. Contributions in this strand of literature showed clearly that even in drastically simple formulations, the possible optimal dynamics of an economy submitted to climate impacts could be very complex over time. Another reason is that, contrarily to the non renewable depletion problem, a problem which is largely under control of economic decisions concerning the rate of exploitation of natural resources, the climate change problem is only indirectly under economic control, mainly through the mitigation of carbon emissions. The main drivers of it are dependent on uncontrolled physical and biological processes. A third reason is the significant uncertainties affecting the future climate, casting doubts about the relevance of deterministic approaches like the ones popularized by the RES symposium for the long run analysis of the consequences of the depletion of scarce non renewable resources.

Most analysis of the pollution accumulation problem stem from the tradition pioneered by Forster (1975) of introducing a pollution dependent social welfare function into a conventional optimal growth model, the work of Krautkraemer (1985) being a prominent example of this approach. The Forster framework is a polluting growth model, that is output generation is responsible for pollution accumulation inside the environment. As remarked by Farzin (1993), carbon accumulation involves the burning of fossil fuels, hence this is the use of some specific inputs which is responsible for the climate problem. It involves a slightly different modeling approach, the polluting resources framework, either exhaustible or inexhaustible. The usual conclusions from such models are that the depletion of fossil fuels should in general be slowed down under an environmental constraint. However no definite conclusions can be derived concerning the trend of capital accumulation and more generally the economy growth of the economy. Of course, imposing an environmental constraint over the economy should result into some overall welfare loss, but the important issue is when and how much loss should be incurred. A rather precise answer to this question can be given for the long run state of the economy (Tahvonen and Kuuluvainen, 1993, Withagen, 1994), but effects along the transition toward this long run state are typically hard to assess. They appear to depend simultaneously of the shape of the environmental damage function, the shape of the utility function and the properties of the production function. The inherent non linearities of these functions may generate complex dynamic patterns for both the pollution stock and the growth of the economy. This is one of the reason explaining the relative small number of theoretical studies on this topic in the literature, as pointed out by Krautkraemer in his survey (1998).

To make progress in this direction, we depart from the usual environmental damage approach of the earlier literature by endorsing an alternative route pioneered by Chakravorty *et al.* (2006) in several recent papers. In their framework, there is no direct damages from the accumulation of carbon into the atmosphere for low carbon concentrations. But be crossed over some critical threshold, earth climate conditions would become catastrophic. In other words, the implicit environmental marginal damage is assumed to be zero up to the carbon concentration threshold and becomes infinite above the threshold. In a deterministic context, the society should stabilize the carbon concentration at most at this security level, and thus satisfy at all time a ceiling constraint over the pollution stock. This approach echoes the current policy proposal of stabilizing the average temperature rise to no more than $+2^0\,C$ by the end of the century, an objective which amounts to target some maximum atmospheric concentration level. The original Chakravorty *et al.* ceiling framework is cast in a partial equilibrium context. We extend this approach to general equilibrium contexts by plugging the ceiling model inside a Dasgupta-Heal-Stiglitz like framework. While satisfying the ceiling constraint, the economy progressively exhausts the fossil fuels reserves while benefiting from the economic growth induced by capital accumulation.

We assume some self-cleaning capacity of the environment. Thus when the economy is constrained by the carbon ceiling, it can at least consume the amount of fossil fuels allowed by the natural regeneration of carbon inside the environment. Pollution accumulation below the ceiling generating no direct environmental cost, the problem is of only interest in a situation where the initial resource and capital endowments would trigger a rise of the pollution stock up to the ceiling level in finite time. On the other hand, the progressive depletion of the polluting resource entails a progressive decline of the resource exploitation rate independently of the carbon problem. Thus, there should exist some finite time such that even without taking into account the carbon pollution problem, the exploitation rate of the resource should fall below the natural regeneration flow when at the ceiling. In the very long run, an economy facing the depletion of the fossil fuels should not be constrained anymore by the carbon problem. The result will be a typical three phases scenario: a first phase of pollution accumulation until the ceiling constraint begins to bind, a temporary phase at the ceiling until the resource is sufficiently depleted to induce a last phase during which the economy is only facing the depletion of its energy primary sources.

As noticed before, in a polluting resource model, this is the use of the resource which is responsible for the pollution problem. Hence, the capital input may be seen as a sort of a 'green' input, its use generating no pollution per itself. Hence capital accumulation and substitution for the use of the polluting resource is both a way to face the depletion of exhaustible resources, like in the classical Dasgupta-Heal-Stiglitz framework, and a device to alleviate the environmental burden of carbon pollution. In this context, we are primarily interested into characterizing the optimal growth path of the economy along the transition and assess the sensitivity of the optimal path to different initial conditions, in particular the severity of the carbon constraint. One main conclusion of the analysis is that the economic growth under a carbon constraint may be of two main types. A first type corresponds to an overall positive growth trend of the economy while the second type correspond to a more or less complex structural adjustment process involving temporary phases of negative economic growth, either before the ceiling begins to bind or either during a time period when the ceiling constraint would be binding. We also provide a thorough analysis of the efficiency requirements of a resource exploitation policy. Contrarily to the usual Hotelling efficiency rule, the rates of returns over the resource and the capital asset should not be anymore equalized under an environmental constraint. We show that the rate of return over the resource should be higher than the rate of return over capital before the ceiling constraint begins to bind and should rise continuously. When reaching the ceiling, the wedge between the rates of return should jump down and may even become negative during a time phase at the ceiling. Most of the characteristics of the optimal growth path follow from this efficiency behavior.

The paper is organized as follows. The next section 2 presents the model and discuss the implications of efficiency at a fairly general level. Section 3 introduces the specified version of the model analyzed in the sequel. The sections 4, 5 and 6 perform the analysis of the optimal path after, during and before the ceiling constraint is active, respectively. The last section 7 concludes.

2 The model

The global economy produces a composite good from labor, man made capital and a polluting non renewable resource (fossil fuels) with a constant returns to scale technology. The global population is assumed to be constant and normalized to one. Inputs productivities benefit from an exogenous trend of technical progress, assumed of the Hicksian type at a rate δ. Let $y(t)$ denote the output level per capita, $K(t)$ be the capital stock per capita and $x(t)$ the resource exploitation rate per capita. Then the production possibilities frontier of the economy in per capita terms is described as: $y(t) = e^{\delta t} f(K(t), x(t))$. $f(K, x)$ is a production function in intensive form describing an integrated production process in the Samuelson sense. $f(.)$ exhibits decreasing returns to scale and is assumed to be increasing in (K, x), concave and such that $f(0, x) = f(K, 0) = 0$, $K \geq 0$ and $x \geq 0$. The output flow of the composite good, $y(t)$, is split between consumption, $c(t)$ and investment, thus $\dot{K}(t) = y(t) - c(t)$ describes the capital accumulation motion over time together with the initial condition: $K(0) = K^0 > 0$, K^0 being given.

Let $X(t)$ be the available natural resource stock at time t, $X(0) = X^0 > 0$ being given, then $\dot{X}(t) = -x(t)$. Fossil fuels burning generates a pollution flow proportional to the resource consumption rate $\zeta x(t)$. The pollution flow accumulates inside the atmosphere. Let $Z(t)$ be the atmospheric carbon stock level at time t. This stock is submitted to some natural self-regeneration process assumed proportional to the carbon stock size for the sake of simplicity. Then the pollution stock law of motion is defined as: $\dot{Z}(t) = \zeta x(t) - \alpha Z(t)$. Carbon pollution does not harm either welfare or the production possibilities of the economy but, be crossed over some critical threshold $\bar{Z}$, earth climate conditions would become catastrophic. The result is a mandated constraint upon admissible carbon concentrations: $Z(t) \leq \bar{Z}$. Assume $Z^0 < \bar{Z}$ to give content to the problem.

The society objective is to maximize a felicity function of the form:

$$\mathcal{U} \;=\; \int_0^\infty u(c(t))e^{-\rho t}dt \;,$$

where $\rho > 0$ is the social discount rate and $u(c)$ is an utility function assumed increasing, strictly concave and satisfying the Inada condition: $\lim_{c\downarrow 0} u'(c) = +\infty$.

An *optimal path* from (K^0, X^0, Z^0) is a vector sequence $\{(c(t), x(t), K(t), X(t), Z(t))$, $t \geq 0\}$ solving the following (OP) program:

$$(OP) \qquad \begin{cases} \max_{\{c(t),x(t)\}} \;\; \mathcal{U} & & \\ & \dot{K}(t) = e^{\delta t} f(K(t), x(t)) - c(t) & K(0) = K^0 \text{ given} \\ & \dot{X}(t) = x(t) & X(0) = X^0 \text{ given} \\ & \dot{Z}(t) = \zeta x(t) - \alpha Z(t) & Z(0) = Z^0 \text{ given} \\ & c(t) \geq 0 \;,\; x(t) \geq 0 & \\ & K(t) \geq 0 \;, X(t) \geq 0 & \\ & \bar{Z} - Z(t) \geq 0 & \end{cases} .$$

Under our assumptions over $f(.)$ and $u(.)$, any solution of the program (OP) verifies $c(t) > 0$, $x(t) > 0$ and $K(t) > 0$, $t \geq 0$. Furthermore $x(t) > 0$ implies that $X(t) > 0$, that is the natural resource stock has to be depleted only asymptotically. In addition $x(t) > 0$ implies that $\dot{Z}(t) > -\alpha Z(t)$, implying in turn that $Z(t) > Z^0 e^{-\alpha t} > 0$. Hence the solution set of the program (OP) may be studied with the following Lagrangian in present value:

$$\begin{aligned} \mathcal{L}(t) &= u(c(t))e^{-\rho t} + \pi(t)\left[e^{\delta t} f(K(t), x(t)) - c(t)\right] - \lambda_X(t)x(t) \\ &\quad -\lambda_Z(t)\left[\zeta x(t) - \alpha Z(t)\right] + \nu(t)\left[\bar{Z} - Z(t)\right] \ . \end{aligned}$$

Dropping time and functional dependency for the ease of reading, a first set of optimality conditions is:

$$u'(c)e^{-\rho t} = \pi \tag{2.1}$$

$$-\frac{\dot{\pi}}{\pi} = e^{\delta t} f_K \ . \tag{2.2}$$

These conditions describe the standard optimal growth rules of a Ramsey-Solow type growth model. Time differentiating (2.1) while using (2.2) results into the usual Ramsey-Keynes condition:

$$-\frac{u''(c)\dot{c}}{u'(c)} + \rho = e^{\delta t} f_K(K, x) \ . \tag{2.3}$$

To these conditions have to be added optimal natural resources and pollution management conditions:

$$\pi e^{\delta t} f_x = \lambda_X + \zeta \lambda_Z \tag{2.4}$$

$$\dot{\lambda}_X = 0 \tag{2.5}$$

$$\dot{\lambda}_Z = \alpha \lambda_Z - \nu \tag{2.6}$$

$$\nu \geq 0 \ , \quad \nu(\bar{Z} - Z) = 0 \ , \quad \bar{Z} - Z \geq 0 \ . \tag{2.7}$$

The condition (2.5) shows that λ_X, the opportunity cost of depleting the natural resource in present value, should be constant along any optimal trajectory, that is the usual Hotelling rule. The complementary slackness condition (2.7) shows that $\nu(t)$ should be nil during any time phase below the ceiling and (2.6) shows in addition that the opportunity cost of pollution, $\lambda_Z(t)$, should grow exponentially at the rate α during such a time phase.

An interesting discussion requires that the ceiling constraint eventually binds during some time phase. During such a time phase, the rate of exploitation of the resource is constrained by the self-regenerating capacity of the environment to a constant level $\bar{x} \equiv \alpha\bar{Z}/\zeta$. Since the exhaustible resource is continuously depleted during a time phase constrained by the ceiling, there should exist some time $\bar{t}$ when even without an atmospheric carbon constraint, the economy should decide a rate of exploitation of the resource and hence a level of emission lower than $\bar{x}$. After $\bar{t}$, the rate of exploitation of the resource should continue to decline, the ceiling constraint never binding anymore. The opportunity cost of carbon pollution should then be nil after $\bar{t}$, that is $\lambda_Z(t) = 0$, $t \geq \bar{t}$. Since $Z^0 < \bar{Z}$ by assumption, the economy is not constrained by the carbon ceiling initially. With a sufficiently high initial resource stock, the optimal exploitation rate of the polluting resource will make increase the atmospheric carbon stock until some time $\underline{t}$ when $Z(\underline{t}) = \bar{Z}$. This justifies to concentrate upon a three time phases scenario: a first pre-ceiling phase $[0, \underline{t})$ during which carbon pollution increases until the carbon cap is attained, a ceiling phase $[\underline{t}, \bar{t})$ during which the exploitation rate of the resource is constrained to the $\bar{x}$ level, and last a post ceiling phase $[\bar{t}, \infty)$ during which the economy is no more constrained by the ceiling and follows the optimal path of the Stiglitz, Dasgupta and Heal original framework.

A complete characterization of the optimal path implies to put more structure on the model fundamentals. Before considering a specified version we conclude this section by a discussion of efficiency in the present context. In a pure resource depletion problem, efficiency, or more precisely technical efficiency, amounts to minimize the cumulated use of the non renewable resource along some given feasible consumption path (Solow, 1974). To express the efficiency conditions having to apply during the different phases, differentiate (2.4) w.r.t. time to obtain:

$$\delta + \frac{\dot{\pi}}{\pi} + \frac{\dot{f}_x}{f_x} = \frac{\zeta\dot{\lambda}_Z}{\lambda_X + \zeta\lambda_Z} \ .$$

Making use of (2.2) this is equivalent to:

$$\frac{(\dot{e^{\delta t} f_x})}{(e^{\delta t} f_x)} - e^{\delta t} f_K = \frac{\zeta\dot{\lambda}_Z}{\lambda_X + \zeta\lambda_Z} \ . \tag{2.8}$$

Since $\lambda_Z(t) = 0$ for $t \geq \bar{t}$, efficiency requires that during the last post-ceiling phase:

$$\frac{(\dot{e^{\delta t} f_x})}{(e^{\delta t} f_x)} = e^{\delta t} f_K \ .$$

This is the well known form of the efficiency condition, or the Hotelling rule for efficient plans. The physical rate of return over the resource appearing in the l.h.s. has to be equalized at any time to the physical rate of return over the capital stock, that is the marginal productivity of capital. Since an efficient policy has to minimize the use of the exhaustible resource, this objective may be achieved through higher investment in capital accumulation. For any feasible consumption path, the efficiency rule states the efficient intertemporal trade-off between the cost in terms of the increased resource use required by a higher rate of capital accumulation and the benefit in terms of future resource savings.

The above reasoning is significantly modified when considering the pollution impact of consuming the natural resource. To the difference of a polluting growth model where this is output production which generates pollution, only the use of the resource is responsible for pollution in a polluting resource model. Hence accumulating more capital is as before a way to save the resource but also a way to reduce future carbon emissions. The consequence is that the physical rates of returns of the resource and man made capital asset should not be equalized anymore. This may be observed by first computing the efficiency condition during the pre-ceiling phase $[0, \underline{t})$. Since $\nu(t) = 0$, $t < \underline{t}$, $\dot{\lambda}_Z = \alpha\lambda_Z$ through (2.6). Denote by:

$$n(t) \equiv \frac{(\dot{e^{\delta t} f_x})}{(e^{\delta t} f_x)} - e^{\delta t} f_K \ .$$

$n(t)$ is the wedge between the physical rates of return of the resource and man made capital. Then (2.8) is equivalent to:

$$n(t) = \alpha \frac{\zeta\lambda_Z}{\lambda_X + \zeta\lambda_Z} \ , \quad t \in [0, \underline{t}) \ . \tag{2.9}$$

Since $\lambda_Z(t) > 0$, it appears that $n(t) > 0$, the physical rate of return over the resource must now be higher than the rate of return over capital to compensate for the pollution effect of burning fossil fuels. Since $\zeta\lambda_Z = e^{\delta t}\pi f_x - \lambda_X$ through (2.4), it results from the above that:

$$n(t) = \alpha \frac{e^{\delta t}\pi f_x - \lambda_X}{e^{\delta t}\pi f_x} = \alpha \left(1 - \frac{\lambda_X}{e^{\delta t}\pi f_x} \right) \ .$$

Time differentiating gets:

$$\begin{aligned}
\dot{n}(t) &= \frac{\alpha\lambda_X}{(e^{\delta t}\pi f_x)^2}\left[\delta e^{\delta t}\pi f_x + e^{\delta t}\pi f_x\left(\frac{\dot{\pi}}{\pi}+\frac{\dot{f}_x}{f_x}\right)\right] \\
&= \frac{\alpha\lambda_X}{e^{\delta t}\pi f_x}\left[\delta + \frac{\dot{f}_x}{f_x} - e^{\delta t} f_K\right] \\
&= \left[\alpha - \alpha\left(1-\frac{\lambda_X}{e^{\delta t}\pi f_x}\right)\right] n(t) \\
&= (\alpha - n(t))\, n(t)\ . \qquad (2.10)
\end{aligned}$$

Making use of the definition of $n(t)$, (2.10) is equivalent to the following efficiency condition having to apply before the atmospheric ceiling constraint begins to be binding:

$$\frac{(e^{\delta t}\dot{f}_x)}{(e^{\delta t} f_x)} + \frac{\frac{d}{dt}\left(\frac{(e^{\delta t}\dot{f}_x)}{(e^{\delta t} f_x)} - e^{\delta t} f_K\right)}{\frac{(e^{\delta t}\dot{f}_x)}{(e^{\delta t} f_x)} - e^{\delta t} f_K} - \alpha = e^{\delta t} f_K\ . \qquad (2.11)$$

Under a pollution ceiling constraint, efficiency requires a positive wedge between the rate of return over the resource and the rate of return over man made capital. From (2.10) one gets also: $\dot{n}(t)/n(t) = \alpha\lambda_X/(e^{\delta t}\pi f_x) > 0$, that is the wedge should increase over time before attaining the ceiling. Turning now to the expression of the wedge during the ceiling phase, $x(t)$ being constant during this phase, $\dot{f}_x/f_x$ is given by $f_{Kx}\dot{K}/f_x + f_{xx}\dot{x}/f_x$ and by $f_{Kx}\dot{K}/f_x$ before and during the ceiling phase. Hence $n(t)$ makes a jump at $\underline{t}$ because of the non differentiability of $x(t)$ at this time. It may even be the case that the wedge turns from positive to negative. This will be in particular the case for the specified economy we are going to examine now.

3 A specified economy

Assume that $f(K,x)$ is of the Cobb-Douglas class:

$$f(K,x) = K^\beta x^\gamma\ ,\ 0<\beta<1\ ,\ 0<\gamma<1\ ,\beta+\gamma<1\ .$$

Furthermore, the utility function is of the CRRA form:

$$u(c) = \frac{1}{1-\eta}c^{1-\eta}, \eta>0\ ,\eta\neq 0\ .$$

Assume in addition that $\beta < 1 < \eta$. The optimal path can be identified thanks to a backward solving strategy. Firstly, concerning the post-ceiling phase, the carbon constraint being no more active, the solution path is simply the optimal path of a Stiglitz (1974) like optimal growth model under an exhaustible resource constraint. The only difference is that the initial level of the resource stock, $\bar{X} \equiv X(\bar{t})$ is endogenous while the initial extraction rate is set to $x(\bar{t}) = \bar{x}$ by continuity of the resource exploitation plan. Secondly, $x(t) = \bar{x}$ during the ceiling phase implies that during this phase, the dynamics of the economy corresponds to a transitory solution of a standard one sector Ramsey-Solow model, the production function being parametrized by $\bar{x}$. Last, only the dynamics of the pre-ceiling phase requires the identification of the motion of the three state variables $(K(t), X(t), Z(t))$.

Let $a \equiv c/K$ and $b \equiv y/K$. Then in the Cobb-Douglas case, $e^{\delta t} f_K = \beta y/K = \beta b$. Denote by $g^h(t) \equiv \dot{h}/h$, the growth rate of any time variable $h(t)$. Making use of the Ramsey Keynes condition (2.3), it is immediately checked that for all time phases:

$$g^c(t) = \frac{1}{\eta}\left(\beta b(t) - \rho\right) \tag{3.1}$$

$$g^K(t) = b(t) - a(t) \tag{3.2}$$

$$g^a(t) = g^c(t) - g^K(t) = a(t) - \frac{\eta - \beta}{\eta} b(t) - \frac{\rho}{\eta} \ . \tag{3.3}$$

In the (a, b) plane, $g^c(t) > 0$ iff $b(t) > \rho/\beta$ and:

$$g^a(t) > 0 \iff b(t) < b^a(a(t)) \equiv \frac{\eta}{\eta - \beta} a(t) - \frac{\rho}{\eta - \beta} \ .$$

The locus $\dot{a} = 0$ is a positive sloping line $b^a(a)$ cutting the horizontal axis at $a = \rho/\eta$ and of slope higher than one. By construction, the locus $\dot{c} = 0$, that is the horizontal $b = \rho/\beta$, cuts the $\dot{a} = 0$ locus along the locus $\dot{K} = 0$, that is the bisectrix $b = a$ (see Figure 1). These dynamic features are phase independent. This is not the case for the laws of motion of $y(t)$, $x(t)$ and $b(t)$ which depend upon the phase dependent efficiency conditions. Let us turn to the study of the successive phases in reverse time order.

4 The post-ceiling phase

During the post-ceiling time phase of infinite duration $[\bar{t}, \infty)$, efficiency requires that $(\dot{e^{\delta t} f_x})/(e^{\delta t} f_x) = e^{\delta t} f_K$. Straightforward computations show that this implies the following motions of $x(t)$, $y(t)$ and $b(t)$:

$$
\begin{aligned}
g^x(t) &= \frac{1}{1-\gamma}\,(\delta - \beta a(t)) && (4.1)\\
g^y(t) &= \beta b(t) - \frac{\beta}{1-\gamma} a(t) + \frac{\delta}{1-\gamma} && (4.2)\\
g^b(t) &= \frac{1-\beta-\gamma}{1-\gamma} a(t) - (1-\beta) b(t) + \frac{\delta}{1-\gamma}\ . && (4.3)
\end{aligned}
$$

It results from (4.3) that:

$$
g^b(t) > 0 \iff b(t) < b_3^b(a(t)) \equiv \frac{1-\beta-\gamma}{(1-\beta)(1-\gamma)} a(t) + \frac{\delta}{(1-\beta)(1-\gamma)}\ .
$$

The slope of the line $b_3^b(a)$ corresponding to the locus $\dot{b} = 0$ is positive and lower than one and $b_3^b(0) = \delta/(1-\beta)(1-\gamma) > 0$. The loci $\dot{a} = 0$ and $\dot{b} = 0$ intersect at (a^*, b^*), the unique stationary state of the $(a(t), b(t))$ dynamical system.

$$
\begin{aligned}
a^* &= \frac{\rho(1-\beta)(1-\gamma) + \delta(\eta - \beta)}{\beta(1-\beta-\gamma+\gamma\eta)} && (4.4)\\
b^* &= \frac{\rho(1-\beta-\gamma) + \delta\eta}{\beta(1-\beta-\gamma+\gamma\eta)}\ . && (4.5)
\end{aligned}
$$

The expressions of the asymptotic growth rates of K, y and c when the economy approaches the steady state (a^*, b^*) can also be easily computed. Since $c(t)/K(t) \to a^*$, $g^c(t) \to g^K(t)$ and since $y(t)/K(t) \to b^*$, $g^y(t) \to g^K(t)$. Hence denoting by g^* the common level of the asymptotic growth rates:

$$
g^* = \frac{1}{\eta}\,(\beta b^* - \rho) = \frac{\delta - \gamma\rho}{1-\beta-\gamma+\gamma\eta}\ . \qquad (4.6)
$$

$g^* > 0$ iff $\rho < \delta/\gamma$, which is the expression of the Stiglitz survival condition in the present model. The economy has to be sufficiently patient with respect to the speed of technical progress to be able to sustain a positive steady growth path of its main macroeconomic variables in the very long run. Assume

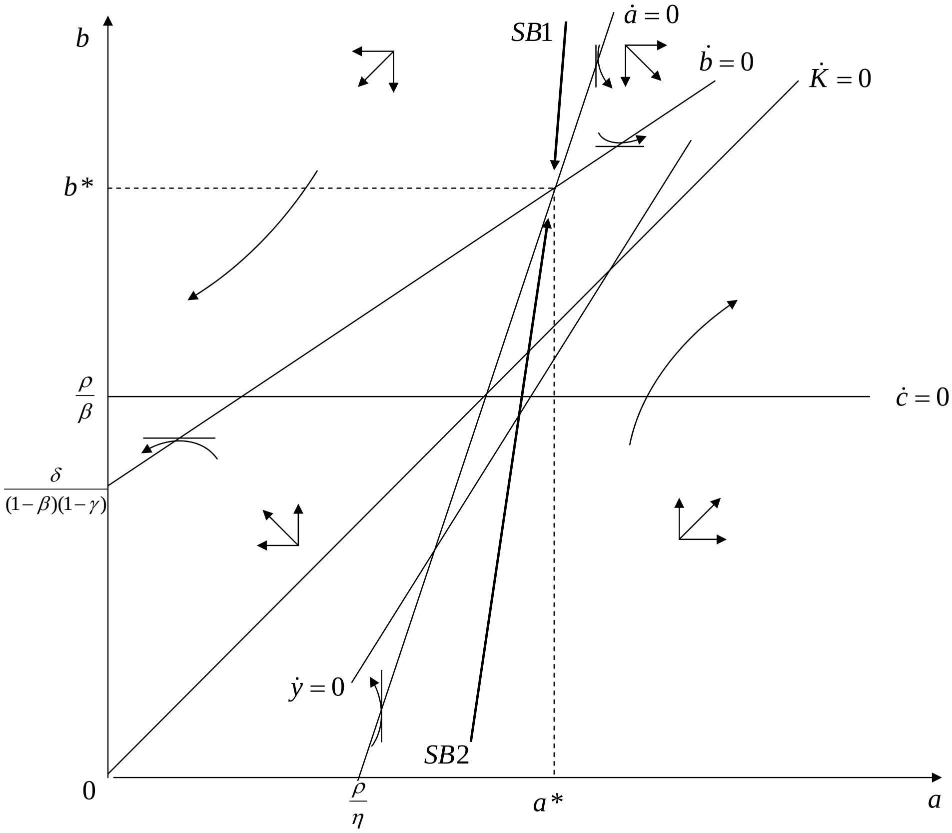

Figure 1: **Optimal growth after the ceiling if** $\gamma\rho < \delta$

that this condition holds. Then the optimal path corresponds to the saddle branches labeled $SB1$ and $SB2$ on Figure 1.

Since (a^*, b^*) are located above the loci $\dot{c} = 0$, $\dot{K} = 0$ and $\dot{y} = 0$, the optimal path along $SB1$ corresponds to a positive growth of all the economy macroeconomic variables, the consumption level, the capital stock level and the output level. By contrast, $SB2$ corresponds to a complex capital desaccumulation pattern where the consumption rate, the output level and the capital stock level may temporarily decrease before having to increase when the economy converges toward the steady state.

Since we are primarily interested into the description of the optimal policy under an environmental constraint, that is during the pre-ceiling and the ceiling phase, we choose to focus only upon scenarios where the economy follows the $SB1$ path after the end of the ceiling period.

Now note that (4.4) implies that $a^* > \delta/\beta$ since $\eta > 1$ by assumption.

$\dot{a}(t)$ being strictly negative along $SB1$, this implies in turn that $a(t) > \delta/\beta$, $t \geq \bar{t}$ and thus, taking (4.1) into account, $\dot{x}(t) < 0$. Once the economy leaves the ceiling, it never returns back to it.

Next, denote by $\bar{K} \equiv K(\bar{t})$ the capital stock inherited from the previous time phases for a given $\bar{t}$. With respect to the standard Stiglitz framework, the initial extraction level, $x(\bar{t})$, is here constrained to be given by $\bar{x}$. This defines implicitly $\bar{X}$, the needed resource stock to follow the optimal path from $\bar{K}$ at $\bar{t}$ as a function of $\bar{K}$ and $\bar{x}$. Denote in addition $\bar{c} = c(\bar{t})$, then it can be shown that:

Proposition P. 1 *During the last post-ceiling phase starting from $\bar{K}$ at some given $\bar{t}$:*

1. $\partial\bar{X}/\partial\bar{K} > 0$, $\partial\bar{X}/\partial\bar{x} > 0$.
2. $\partial g^c(t)/\partial\bar{K} < 0$, $\partial g^c(t)/\partial\bar{x} > 0$, $t \geq \bar{t}$.
3. $\partial\bar{c}/\partial\bar{K} > 0$, $\partial\bar{c}/\partial\bar{x} > 0$.
4. $\partial g^x(t)/\partial\bar{K} > 0$, $\partial g^x(t)/\partial\bar{x} < 0$, $t \geq \bar{t}$.
5. $\partial g^K(t)/\partial\bar{K} < 0$, $\partial g^K(t)/\partial\bar{x} > 0$, $t \geq \bar{t}$.

Proof : See Appendix A.1.

Claim 1 is the main result of the proposition. It shows first the existence of a positive relationship between the amount of resource needed to follow an optimal policy once the economy can escape the atmospheric carbon ceiling constraint and the amount of accumulated capital. In other words, accumulating more capital before the end of the ceiling phase does not alleviate the need to keep a sufficient amount of resource reserves. Secondly, a less stringent ceiling constraint, that is a higher $\bar{x}$, implies a higher available resource amount after the ceiling phase. This requires some natural resource savings before, either in the form of a reduced extraction rate before attaining the ceiling or through a shorter time at the ceiling. The remaining claims are usual in Ramsey-Solow growth models. A higher capital stock $\bar{K}$

at the beginning of the time phase means higher consumption rates at all time together with a slower consumption growth and a reduced investment rate. Similar conclusions arise when the ceiling constraint is relaxed.

5 The ceiling phase

The motivation for a thorough study of the ceiling phase is two-fold. Firstly, since the economy is constrained to use the resource at the rate $\bar{x}$, the only way to sustain growth during this phase is through capital accumulation. The economy has to follow some transitory Ramsey-Solow growth path with production possibilities parametrized by $\bar{x}$. But it may be the case that optimal capital accumulation before $\underline{t}$, an investment process intended to delay the attainment of the ceiling, results in the need to reduce the capital stock during the ceiling phase. Hence depending upon inherited conditions concerning the previously accumulated capital stock and the remaining resource stock at the beginning of the ceiling phase, the economy can experience complex patterns of evolution of its main macroeconomic variables. Secondly, assume that the polluting resource is a renewable one instead of an exhaustible one and that the availability of this renewable resource is higher than $\bar{x}$. Then the economy would remain constrained forever by the ceiling after $\underline{t}$ and the ceiling phase would be an infinite duration terminal phase. Such an outcome is a limit situation where the exhaustibility of fossil fuels could be somewhat neglected.

Since $b(t) = e^{\delta t} K(t)^{\beta-1} \bar{x}^{\gamma}$ during the ceiling phase, the dynamics of $b(t)$ is now given by:

$$g^b(t) = (1-\beta)a(t) - (1-\beta)b(t) + \delta \ . \tag{5.1}$$

This implies that $\dot{b} > 0$ iff $b < a + \delta/(1-\beta) \equiv b_2^b(a)$. The lines $b^a(a)$ and $b_2^b(a)$ cross at $(\hat{a}, \hat{b})$ defined by:

$$\hat{a} = \frac{(1-\beta)\rho + (\eta-\beta)\delta}{\beta(1-\beta)} \tag{5.2}$$

$$\hat{b} = \frac{(1-\beta)\rho + \delta\eta}{\beta(1-\beta)} \ . \tag{5.3}$$

It is immediately checked that $a^* < \hat{a}$ and $b^* < \hat{b}$. $(\hat{a}, \hat{b})$ is the long run steady state toward which the economy would converge if the polluting resource was renewable instead of being non renewable. Output, consumption and capital accumulation growth rates would converge toward the asymptotic common level $\hat{g} = \delta/(1-\beta)$. The optimal policy would be to follow the stable manifolds $S\hat{B}_1$ or $S\hat{B}_2$ (see Figure 2).

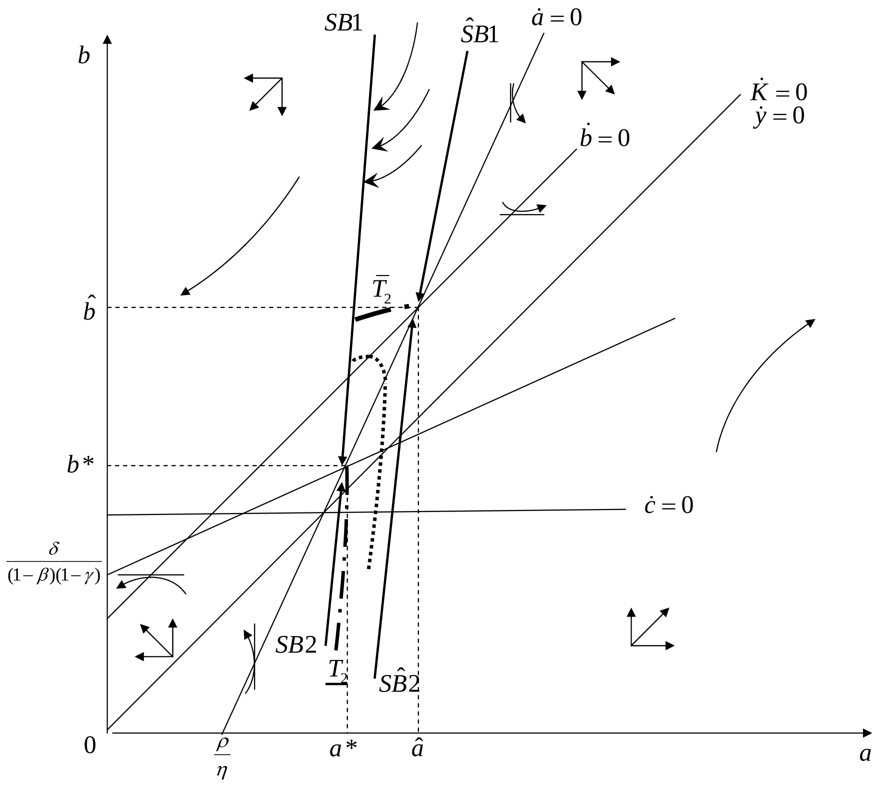

Figure 2: **Optimal growth during the ceiling**

The manifold $S\hat{B}_1$ corresponds to a positive growth path of consumption , output and the capital stock. $S\hat{B}_2$ shows more subtle dynamics (see Figure 2). Let $\underline{a} \equiv a(\underline{t})$ and $\underline{b} \equiv b(\underline{t})$. For sufficiently low $(\underline{a}, \underline{b})$ inherited from the pre-ceiling phase, the economy experiences first a consumption decline together with an output decline and a negative investment rate (the capital stock is also diminishing through time). Next the conversion of the capital stock into consumption good (remember that the capital accumulation process is perfectly reversible in the model under consideration) allows for a positive growth of the consumption rate despite a declining output rate. Whence the locus $\dot{K} = \dot{y} = 0$ has been attained, the economy reverts to a positive growth regime of consumption, output and the capital stock. This shows that even without taking into consideration the exhaustible character of fossil fuels, the dynamics of an economy forever constrained to mitigate carbon emissions up to some constant level $\bar{x}$ compatible with the self-cleaning capacities of the environment can be quite complex.

Turn back to the original problem. The optimal trajectory during the ceiling phase has to connect in finite time to some $(\bar{a}, \bar{b})$ located along the SB_1 saddle-branch for the post-ceiling phase, since we have chosen to consider only this type of post-ceiling dynamics. This requires that such a trajectory initiates from some $(\underline{a}, \underline{b})$ located in between the SB_1-SB_2 manifold and the $S\hat{B}_1$-$S\hat{B}_2$ manifold. These trajectories may be of two types. The type I trajectories initiate from above the trajectory emanating from $(\hat{a}, \hat{b})$ in the south-west direction (a trajectory labeled $\bar{T}_2$ on Figure 2). Both $a(t)$ and $b(t)$ decrease over time along type I trajectories and the economy follows a positive growth path of its main macroeconomic variables. The fact that $a(t)$ and $b(t)$ decrease shows that the growth rates of consumption and output should be lower than the growth rate of the capital stock. The type II trajectories initiate from below $\bar{T}_2$. For a sufficiently low initial $(\underline{a}, \underline{b})$, the optimal path under the ceiling constraint may be composed of at most five successive phases.

During a first phase $[\underline{t}, t_c)$, the consumption rates and the output rate decline while some fraction of the capital stock is converted to consumption. At t_c, the consumption rate reaches its minimum. Next during a second time phase $[t_c, t_K)$, the consumption rate increases while both the output level and the capital stock size continue to decrease. Since $b(t)$ increases during such a phase, the output level declines at a lower rate than the capital stock. At t_K the capital stock size attains its minimum. During a third time interval $[t_K, t_a)$, the consumption rate, the output rate and the capital stock simultaneously increase. Since $a(t)$ and $b(t)$ increase, the growth rates of consumption and output have to be larger than the growth rate of the capital stock. During a fourth phase $[t_a, t_b)$, the economy continues to expand despite the ceiling constraint but now since $a(t)$ decreases through time, the consumption growth rate becomes lower than the capital stock growth rate. This does not apply to $y(t)$ since $b(t)$ being increasing, $g^y(t) > g^K(t)$. Last, during the fifth phase $[t_b, \bar{t})$, $a(t)$ and $b(t)$ being both decreasing, the economy is permanently growing at a positive rate but both $g^c(t) < g_K(t)$ and $g^y(t) < g^K(t)$.

It remains to show that for a given $\underline{t}$, starting from some given initial endowments $(\underline{K}, \underline{X})$ in man made capital and resource assets, there exists a unique optimal path over the ceiling phase $[\underline{t}, \bar{t})$ and the post ceiling phase $[\bar{t}, \infty)$ together with a unique $\bar{t}$, the end time of the ceiling phase. This may be achieved by a fixed point argument in the $(\underline{a}, \bar{t})$ plane. The argument runs as follows.

Consider first type I trajectories. $\underline{b} = e^{\delta \underline{t}} \underline{K}^{\beta-1} \bar{x}^{\gamma}$ defines $\underline{b} \equiv b(\underline{t})$ for a given $(\underline{K}, \underline{t})$. In the renewable resource case, $\underline{a} \equiv a(\underline{t})$ is then defined by $b_{\hat{SB}}(\underline{a}) = \underline{b}$, where $b_{\hat{SB}}$ is the implicit relationship between a and b along the $\hat{SB}$ manifold. In the non renewable case, $\underline{b}$ defines an interval for the possible values of $\underline{a}$, $\mathcal{A}(\underline{b}) \equiv [\underline{a}_0(\underline{b}), \underline{a}_1(\underline{b}))$. Let $b_{SB}(a)$ be the implicit relation between a and b along the SB manifold. Note that $b_{SB}(a)$ is an increasing function of a. $\underline{a}_0$ is defined as the solution of $b_{SB}(a) = \underline{b}$. This corresponds to an immediate transition toward the post ceiling phase, that is $\underline{t} = \bar{t}$. $\underline{a}_1$ is defined as the solution of $b_{\hat{SB}}(a) = \underline{b}$. This would imply no convergence toward the SB manifold, that is $\bar{t} \to \infty$ when $\underline{a} \to \underline{a}_1$. Next consider two connecting trajectories $\mathcal{T}$ and $\mathcal{T}'$ initiated from $(\underline{a}, \underline{b})$ and $(\underline{a}', \underline{b})$ respectively, with $\underline{a} < \underline{a}'$. The trajectories not crossing themselves in the phase plane (a, b), the trajectory $\mathcal{T}$ should be located above the trajectory $\mathcal{T}'$. This implies endpoints $(\bar{a}, \bar{b})$ and $(\bar{a}', \bar{b}')$ of $\mathcal{T}$ and $\mathcal{T}'$ respectively on SB_1 such that $\bar{a}' < \bar{a}$ and $\bar{b}' < \bar{b}$. Since the trajectory $\mathcal{T}'$ is located below the trajectory $\mathcal{T}$, its is also located nearer the $\hat{SB}_1$ manifold than the trajectory $\mathcal{T}$. Through a well known property of phase planes, this implies that $(a(t), b(t))$ move slower along the trajectory $\mathcal{T}'$ than along the trajectory $\mathcal{T}$. Since in addition the distance between $\underline{a}'$ and $\bar{a}$ together with the distance between $\underline{b}$ and $\bar{b}'$ is larger than the distances between $\underline{a}$ and $\bar{a}$ or between $\underline{b}$ and $\bar{b}$, it appears that $\bar{t}$ is larger for the trajectory $\mathcal{T}'$ than for the trajectory $\mathcal{T}$. In other words, the geometry of the phase plane defines an increasing relationship between $\underline{a}$ and $\bar{t}$ inside $\mathcal{A}(\underline{b})$, we denote by $\bar{t}_0(\underline{a})$. It is such that $\bar{t}_0(\underline{a}_0) = \underline{t}$ and $\lim_{\underline{a} \uparrow \underline{a}_1} \bar{t}_0(\underline{a}) = +\infty$.

The stock constraint:

$$\underline{X} \quad = \quad \bar{x}(\bar{t} - \underline{t}) + \bar{X}(\bar{K}) \ ,$$

defines another relation between $\bar{t}$ and $\underline{a}$ for a given $(\underline{X}, \underline{K})$ and a given $\underline{K}$. Since $\bar{b} = e^{\delta \bar{t}} \bar{K}^{\beta-1} \bar{x}^{\gamma}$ one gets through the differentiation of the stock condition:

$$\left(\bar{x} + \frac{\delta}{1-\beta}\bar{K}\right) d\bar{t} \;\; = \;\; \bar{X}'(\bar{K}) \frac{\bar{K}}{1-\beta} \frac{d\bar{b}}{\bar{b}}$$

Since $\bar{b}$ is a decreasing function of $\underline{a}$, as noted before, this shows the existence of a decreasing relationship between $\bar{t}$ and $\underline{a}$, we denote by $\bar{t}_1(\underline{a})$. The domain of definition of $\bar{t}_1(\underline{a})$ has to be identified. The stock constraint implies that $\bar{X}$ is confined between two bounds. A lower bound is $\bar{X}_0 \equiv \bar{X}(\underline{K})$. It corresponds to a limit path at the ceiling during which no capital would be accumulated (remember that there is non disinvestment along type I trajectories) and thus to the largest possible value of $\bar{t}$ for a given $\underline{X} > \bar{X}_0$, a value denoted by $\bar{t}_1$. Since in such a case the corresponding $\bar{b}_0$ would be given by $e^{\delta(\bar{t}-\underline{t})}\underline{b}$, $\bar{a}_0$ defined by $b_{SB}(\bar{a}_0) = \bar{b}_0$ should be strictly higher than $\underline{a}_0$ and thus $\underline{a}_1$ solution of $\bar{t}_1(a) = \bar{t}_1$ being strictly higher than $\bar{a}_0$, $\underline{a}_1 > \underline{a}_0$. The upper bound for $\bar{X}$ is trivially $\underline{X}$. This corresponds to an immediate transition toward the post ceiling phase, that is $\bar{t} = \underline{t}$. To this critical $\bar{X}$ corresponds some critical $\bar{K}_1 > \underline{K}$ and thus a critical $\bar{b}_1 < \underline{b}$. $\bar{b}$ being a decreasing function of $\underline{a}$, we conclude that the corresponding $\underline{a}$ should be higher than $\underline{a}_0$. This shows that the curves $\bar{t}_0(\underline{a})$ and $\bar{t}_1(\underline{a})$ intersect at some unique $(\bar{t}^*, \underline{a}^*)$ in the $(\underline{a}, \bar{t})$ plane, defining the optimal end time of the ceiling phase and the optimal level of initial consumption rate $\underline{c}^* = \underline{a}^*\underline{K}$. The connecting trajectory in the (a, b) plane being determined by $(\underline{a}^*, \underline{b})$, the paths of $c(t)$, $y(t)$, $K(t)$ are determined and hence $\bar{K} = K(\bar{t}^*)$ is also determined, giving the optimal level of resource endowment, $\bar{X}^*$, needed to follow the optimal path after the ceiling phase.

Turning to type II trajectories, the temporal implications of the phase plane geometry are roughly the same. For a given $\underline{b}$, to increasing levels of $\underline{a}$ correspond trajectories located nearer the $\hat{SB}$ manifold, hence a slower move along such trajectories over some interval $[\underline{b}, b_1)$. It may also be observed that $\bar{b}$ is now an increasing function of $\underline{a}$, thus the interval $[\underline{b}, \bar{b})$ is enlarged when $\underline{a}$ is increased. Hence there exists an increasing relationship between $\underline{a}$ and $\bar{t}$. If $\underline{b} > b^*$, there exists a value of $\underline{a}$ corresponding to an immediate transition toward the post ceiling phase, thus $\underline{t}$ is the lower bound for possible values of $\bar{t}$ while, as before, $\bar{t}$ tends to infinity when $\underline{a} \to \underline{a}_1$. If $\underline{b} < b^*$, the trajectories connecting to SB_1, the ones which we have chosen to consider, must be located right to the trajectory $\underline{T}_2$, the trajectory converging in finite time toward (a^*, b^*) during the ceiling phase (see Figure 2). Let $\bar{t}_0(\underline{b})$ be the time needed to connect to (a^*, b^*) along this trajectory. Then $\bar{t} \geq \bar{t}_0$ for the family of post-ceiling scenarios under examination. It results that $\bar{t}$ belongs now to the interval $[\bar{t}_0, \infty)$.

If $\underline{b} > b^*$, there exists a critical value of $\underline{a}$, $\tilde{a}(\underline{b})$ defined by $b^a(a) = \underline{b}$. For $\underline{a} < \tilde{a}$, $\bar{b}$ is a decreasing function of $\underline{a}$, hence the stock constraint defines an decreasing relationship between $\underline{a}$ and $\bar{t}$. For $\underline{a} > \tilde{a}$, $\bar{b}$ becomes an increasing function of $\underline{a}$ and thus the stock constraint defines an increasing relationship between $\underline{a}$ and $\bar{t}$. Since $\dot{K} > 0$ along a type II trajectory initiated from $\underline{b} > b^*$, $\underline{K} = \bar{K}$ defines as before the upper length of the ceiling phase, and let T be the highest corresponding value of $\bar{t}$. Then the curve $\bar{t}_1(\underline{a})$ being first decreasing and then increasing while remaining bounded from above by $\bar{T}$ must cross at least once the curve $\bar{t}_0(\underline{a})$. If $\underline{b} < b^*$, $\bar{b}$ is an increasing function of $\underline{a}$, resulting into an increasing relationship between $\bar{t}$ and $\underline{a}$ through the stock constraint. Now the capital stock may be decreasing through time initially. The minimum capital level is attained when the trajectory initiated from $\underline{a}$ intersects the locus $\dot{K} = 0$ in the phase plane. It is immediately verified that to this intersection corresponds a critical increasing function $\tilde{b}(\underline{a})$. This function defines thus a decreasing relationship between $\underline{a}$ and $\tilde{K}$. The stock constraint then defines an upper bound over $\bar{t}$ defined by $\underline{X} = \bar{x}(\bar{t} - \underline{t}) + \bar{X}(\tilde{K}(\underline{a}))$. Let $\tilde{T}(\underline{a})$ this upper bound as a function of $\underline{a}$. $\tilde{T}$ increases with $\underline{a}$ between finite bounds. Since $t_1(\underline{a}) \leq \tilde{T}(\underline{a})$, we conclude that the curves $\bar{t}_0(\underline{a})$ and $\bar{t}_1(\underline{a})$ should cross at least once.

The sensitivity analysis of type II trajectories is rather cumbersome but it is possible to derive some results for the simpler case of type I trajectories. The following proposition summarizes the sensitivity of the optimal growth path to changes in the initial conditions over $\underline{K}$, $\underline{X}$ and $\bar{x}$ for this type of trajectories.

Proposition P. 2 *For type I trajectories:*

1. *A higher initial resource stock level $\underline{X}$ induces:*
   - *A longer ceiling phase, $\partial\bar{t}/\partial\underline{X} > 0$;*
   - *A higher initial consumption level, $\partial\underline{c}/\partial\underline{X} > 0$;*
   - *A lower consumption growth rate, $\partial g^c(t)/\partial\underline{X} < 0$;*

2. *A higher initial capital stock level $\underline{K}$ induces:*
   - *A shorter ceiling phase, $\partial\bar{t}/\partial\underline{K} < 0$;*
   - *A higher initial consumption rate, $\partial\underline{c}/\partial\underline{K} > 0$,*
   - *A lower consumption growth rate, $\partial g^c(t)/\partial\underline{K} < 0$*
3. *A less stringent ceiling constraint, that is a higher level of $\bar{x}$, induces:*
   - *A shorter ceiling phase, $\partial\bar{t}/\partial\bar{x} < 0$;*
   - *Ambiguous effects over the initial consumption rate and the initial consumption growth rate, these two rates being affected in the reverse sense.*

Proof: See Appendix A.2.

As expected, a less stringent ceiling constraint means a shorter stay at the ceiling. This is a direct consequence of the fact that a higher $\bar{x}$ corresponds to a higher resource exploitation rate at the beginning of the ceiling phase and thus a higher requirement in terms of available resource stock to follow the optimal path during the last post-ceiling phase (remember that $\partial\bar{X}/\partial\bar{x} > 0$ as shown in Proposition P.1). Since a less stringent constraint allows in addition for a higher resource consumption rate during the ceiling phase, the conclusion that this phase is shortened follows immediately. However, the consequences of a less stringent constraint over the consumption rate, and thus over the capital accumulation rate during the ceiling phase is generally ambiguous, even for the simpler type I trajectories. The implications of either a higher initial resource endowment, $\underline{X}$, or a higher initial capital stock endowment, $\underline{K}$, are rather straightforward.

We conclude this section by a discussion of the dynamics of the wedge between the physical rates of returns over the man made capital and resource asset during the ceiling phase. It has been already shown that the wedge, $n(t) \equiv (\dot{\overline{e^{\delta t} f_x}})/(e^{\delta t} f_x) - e^{\delta t} f_K$, must be positive and increasing through time before the atmospheric ceiling constraint begins to be binding. What happens to the wedge during the ceiling phase depends upon whether the optimal trajectory $\{a^*(t), b^*(t)\}$ is of type I or II. Since $x(t) = \bar{x}$ during the ceiling phase, $\dot{f}_x/f_x = \beta g^K$, furthermore remember that $f_K = \beta b$. It results that:

$$n(t) \;=\; \delta + \beta g^K(t) - \beta b(t) = \delta - \beta a(t) \;\;,\;\; t \in [\underline{t}, \bar{t})$$

This implies that $\dot{n}(t) = -\beta\dot{a}(t)$, $a(t)$ and $n(t)$ evolve in the reverse direction. In the case of type I trajectories, first $\dot{a}(t) < 0$ and second $a(t) > a^* > \delta/\beta$. Thus $n(t) < 0$ and $\dot{n}(t) > 0$ during the ceiling phase. When attaining the ceiling, that is at $\underline{t}$, the wedge $n(t)$ which was strictly positive and increasing before $\underline{t}$ has to jump down abruptly at $\underline{t}$ and become negative. After this downward jump, the wedge, while remaining negative, increases until $\bar{t}$ when $n(\bar{t}) = 0$, this terminal value being resulting from the Hotelling rule for efficient plans having to apply over the last post-ceiling phase. This is a consequence of the non differentiability of the exploitation path $x(t)$, jumping from a decreasing pattern before $\underline{t}$ to a constant pattern when at the ceiling.

Type II trajectories exhibit more complex patterns as may be expected. If $\underline{b} > b^*$ nothing is changed to the downward jump of the wedge with respect to the case of type I trajectories. The only possible qualitative difference is the possibility of a first time phase during which $\dot{a}(t) > 0$ and thus $\dot{n}(t) < 0$. The wedge may continue to increase in absolute terms before having to decline toward zero at the end of the ceiling phase. But if $\underline{b} < b^*$, it may be the case that $\underline{a}^* < \delta/\beta$ and since it has been shown that initially $\dot{a} > 0$ before changing sign when approaching the end of the ceiling phase, $\bar{t}$, the wedge can remain positive initially, decrease and become temporarily negative before having to increase from a negative level to zero.

It is not easy to supply a straightforward explanation for this complex time behavior. Since $n(t)$ and $a(t)$ move in parallel, the fluctuations of the wedge between the rates of return over the resource and man made capital asset result from the complex adjustment process of the consumption rate and the capital stock when the economy inherits from a large capital stock, $\underline{K}$, at the beginning of the ceiling phase. To conclude, it is worth noting that the jump in the wedge between the rates of return is not a consequence of the exhaustible or not character of the natural resource. If fossil fuels were inexhaustible, $\lambda_X = 0$ before $\underline{t}$ would imply that $n(t) = \alpha$, $t < \underline{t}$, that is the wedge has to be adjusted to the rate of natural self-regeneration of the atmospheric carbon stock. For type I trajectories, the connection will be on the $\hat{SB}_1$ saddle branch, where first $a(t) > \hat{a} > a^* > \delta/\beta$, implying that $n(t)$ should be negative during the last ceiling phase, and second, $\dot{a}(t) < 0$, implying that $n(t)$ should increase over time. For type II trajectories connecting to the $\hat{SB}_2$ saddle branch, depending on either $\underline{a}$ is higher or lower than δ/β, $n(t)$ jumps to a negative or to a positive level and decreases in all cases thereafter, since $\dot{a}(t) > 0$ along this saddle branch.

6 The pre-ceiling phase

The growth dynamics are now solution of a three dimensional system in (a, b, n). Fortunately the autonomous dynamics of $n(t)$ simplifies a lot the analysis which may be at least qualitatively done in the phase plane (a, b). Making use of (2.9), we get the expression of the growth rate of $x(t)$:

$$g^x(t) = \frac{\delta - \beta a(t) - n(t)}{1 - \gamma} \,. \tag{6.1}$$

This gives the expressions of the growth rates of output and $b(t)$ before the ceiling phase:

$$g^y(t) = \beta b(t) - \frac{\beta}{1-\gamma} a(t) + \frac{\delta - n(t)}{1-\gamma} \tag{6.2}$$

$$g^b(t) = \frac{1-\beta-\gamma}{1-\gamma} a(t) - (1-\beta) b(t) + \frac{\delta - \gamma n(t)}{1-\gamma} \,. \tag{6.3}$$

The dynamics of $n(t)$ is described by (2.10) as a simple Ricatti differential equation with general solution:

$$n(t) = \frac{\alpha n_0}{(\alpha - n_0) e^{-\alpha t} + n_0} \qquad n(0) = n_0 \tag{6.4}$$

Remember that $\dot{n}(t) > 0$ during the pre-ceiling phase. Denote by $n(t, n_0)$ the expression (6.4) of $n(t)$ for a particular value n_0 of $n(0)$. This implies that the $\dot{b} = 0$ locus defined by:

$$b = b_1^b(a) \equiv \frac{1-\beta-\gamma}{(1-\beta)(1-\gamma)} a + \frac{\delta - \gamma n(t, n_0)}{(1-\beta)(1-\gamma)}$$

translates downwards over time since $n(t)$ increases. Furthermore, $n(t) > 0$ implies that $b_1^b(a) < b_3^b(a)$, hence above the line $b_3^b(a)$ and to the left of the locus $\dot{a} = 0$, which is independent from $n(t)$, $\dot{a} < 0$ and $\dot{b} < 0$.

There exist two main families of trajectories in the (a, b) plane connecting to trajectories for the ceiling phase, call them type (i) and type (ii) trajectories (see Figure 3). The type (i) family is composed of trajectories initiated to the left of the $\dot{a} = 0$ isocline and above the $b_3(a)$ line. These trajectories are then such that $\dot{a} < 0$, $\dot{b} < 0$ together with $\dot{c} > 0$, $\dot{y} > 0$ and $\dot{K} > 0$. Thus they correspond to a positive growth motion of the economy over time where the capital stock increases at a higher rate than output and the consumption rates. The type (ii) family is composed of trajectories initiated to the right of the $\dot{a} = 0$ isocline and either above or below the $\dot{b} = 0$ isocline. They exhibit a more complex dynamic pattern with possible temporary segments of decreasing consumption and output rates combined with disinvestment in man made capital.

Remembering that type I trajectories during the ceiling phase are located above the line $b_3^b(a)$, it is immediate that trajectories during the pre-ceiling phase connecting to this type of ceiling trajectories belong to the type (i) family. Next, consider the connection toward type II trajectories followed during the ceiling phase. Denote $a_0 \equiv a(0)$ and $b_0 \equiv b(0)$. If $b_0 > b^*$, the connecting trajectories may be either of types (i) or (ii). Type (i) trajectories initiate from (a_0, b_0) such that $b_0 > b^a(a_0)$, that is above the locus $\dot{a} = 0$. Thus, they connect to a type II ceiling trajectory also characterized by $\dot{a} < 0$ and $\dot{b} < 0$. By contrast type (ii) trajectories initiate to the right of the $\dot{a} = 0$ isocline, hence $\dot{a} > 0$ initially for such trajectories. $b(t)$ may then be either temporarily increasing or decreasing depending upon the motion of the $\dot{b} = 0$ locus. In all cases, the trajectory connects on a type II trajectory characterized by $\dot{a} > 0$. If $b_0 < b^*$, connecting trajectories only belong to the type (ii) family. But now, for sufficiently low levels of a_0 and b_0, it may be possible that $\dot{c} < 0$, $\dot{y} < 0$ or $\dot{K} < 0$, that is the economy may experience a negative growth phase before attaining the ceiling.

Having described the qualitative evolutions of $(a(t), b(t), n(t))$ before the ceiling phase it remains to compute the optimal trajectory of the economy starting from some given bundle of initial capital, natural resource and pollution stocks (K^0, X^0, Z^0). This can be achieved through the following procedure. We have shown before that to any given vector $(\underline{K}, \underline{X}, \underline{t})$ of the capital and resource stocks at the beginning of the ceiling phase and any given time arrival at the ceiling, $\underline{t}$, we can associate a unique pair $(\underline{a}, \underline{b})$ and a unique end time of the ceiling period $\bar{t}$. Let us consider some arbitrary n_0. Then this defines through (3.3) and (6.3) a unique $\{a_1(t), b_1(t), n(t)\}$, $t \leq \underline{t}$ trajectory where we denote $a_1(t) \equiv a_1(t, \underline{a}, \underline{b}, \underline{t}, n_0)$ and $b_1(t) \equiv b_1(t, \underline{a}, \underline{b}, \underline{t}, n_0)$. This defines also the growth rate of $x(t)$ before the ceiling: $g^x(t) \equiv g^x(t, \underline{a}, \underline{b}, \underline{t}, n_0)$. From $b_1(0) = (K^0)^{\beta-1} x(0)^\gamma$ we get an expression of $x(0)$ as a function of $(\underline{a}, \underline{b}, \underline{t}, n_0, K^0)$. Since $(\underline{a}, \underline{b})$ are uniquely determined by some $(\underline{K}, \underline{X}, \underline{t}, n_0)$, it thus appears that the vector $(\underline{K}, \underline{X}, \underline{t}, n_0)$ must be a solution of the following system of conditions:

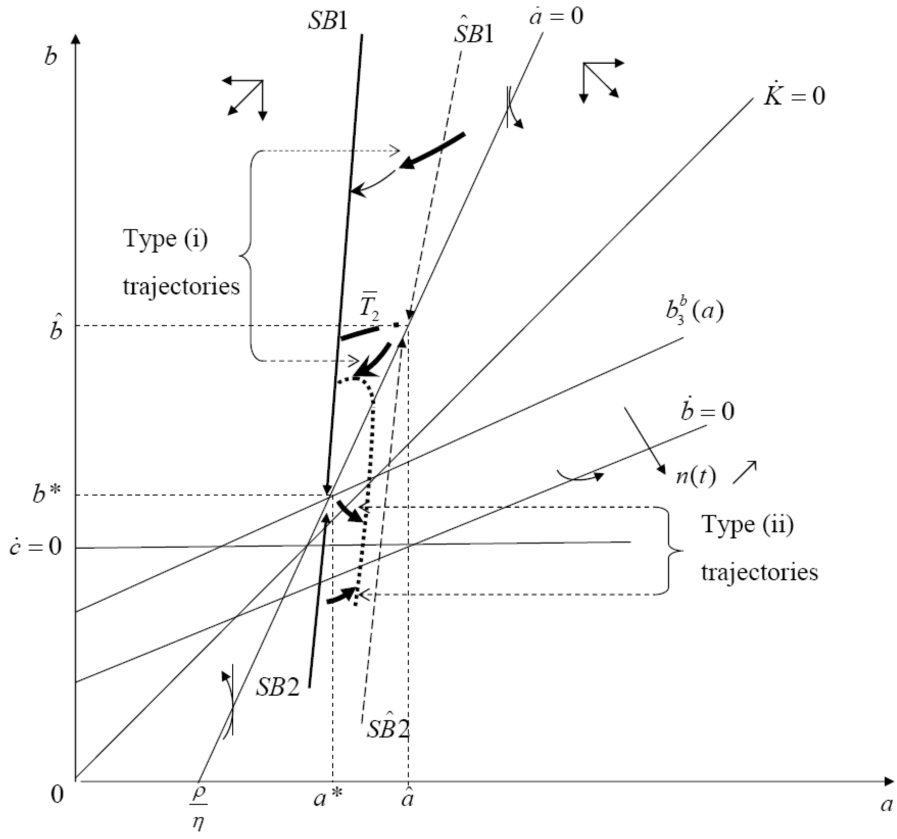

Figure 3: **Optimal growth before the ceiling**

- Continuity condition over the extraction path at $\underline{t}$:

$$\bar{x} \quad = \quad x(0, \underline{K}, \underline{X}, \underline{t}, n_0, K^0)e^{\int_0^{\underline{t}} g^x(t, \underline{K}, \underline{X}, \underline{t}, n_0)dt} \tag{6.5}$$

- Capital accumulation condition before the ceiling:

$$\underline{K} \quad = \quad K^0 e^{\int_0^{\underline{t}} g^K(t, \underline{K}, \underline{X}, \underline{t}, n_0)dt} \tag{6.6}$$

- Resource stock condition:

$$X^0 \quad = \quad x(0, \underline{K}, \underline{X}, \underline{t}, n_0, K^0) \int_0^{\underline{t}} e^{\int_0^t g^x(\tau, \underline{K}, \underline{X}, \underline{t}, n_0)d\tau} dt + \underline{X} \tag{6.7}$$

- Pollution stock condition:

$$\bar{Z}e^{\alpha \underline{t}} \quad = \quad Z^0 + \zeta x(0, \underline{K}, \underline{X}, \underline{t}, n_0) \int_0^{\underline{t}} e^{\int_0^t g^x(\tau, \underline{K}, \underline{X}, \underline{t}, n_0)d\tau} e^{\alpha t} dt \tag{6.8}$$

Summarizing, the economy may follow two main types of evolutions under an environmental constraint taking the form of a cap over admissible carbon concentrations. A first type correspond to a path sequence connecting a type (i) trajectory before $\underline{t}$ to either a type I or a type II trajectory during the ceiling phase. The corresponding optimal macroeconomic scenario involves a positive growth of output, the consumption rate and the capital stock. The second type exhibits more complex dynamics connecting a type (ii) kind of trajectory before $\underline{t}$ to some type II trajectory during the ceiling phase. It is then possible that initially the consumption rate should decline together with the output level and the capital stock size. Such a complex pattern corresponds to a structural adjustment dictated by the severity of the carbon constraint.

7 Conclusion

We have studied the optimal growth dynamics of an economy submitted to a climate constraint. This constraint takes the form of a maximum affordable atmospheric carbon concentration level. This approach departs from the earlier literature relying upon an environmental damages framework to describe the burden of carbon pollution. However, Amigues *et al.* (2011) showed that introducing environmental damages in the present framework does not modify at least qualitatively the main conclusions of the analysis.

One main conclusion of the present study is that tackling the climate issue may require a complex adjustment process involving a temporary decrease of the consumption level. This issue appear to be especially relevant for 'large' initial capital stocks. One could think that such initial conditions have little room in currently observed situations. However, consider the following thought experiment. Assume a growing economy evolving over time without knowledge of the climate problem. Then, such an economy should follow the optimal path of a Dasgupta, Heal, Stiglitz economy, that is the saddle branch we labeled SB_1 in section 4. The result would be a progressive accumulation of capital together with a progressive decline of the fossil fuels reserves. Next assume that the climate change problem is 'discovered' at some time t_0 and that $Z(t_0) < \bar{Z}$. What shows our analysis is that the economy should react to this discovery by jumping to a new position in the phase plane where either

a or b should be decreased. This may be achieved either through a sudden drop down of the consumption rate $c(t)$ and/or through a sudden drop down of the extraction rate $x(t)$. A consumption drop is equivalent of a drop of a at t_0 while a drop of the extraction rate entails a drop down of $y(t)$, that is a drop down of $b(t)$ at t_0. Thus the economy only option is to combine in various ways a consumption rate drop with a resource exploitation rate drop. But the required jump down may be so high that it can be better to initiate a first time phase of declining consumption rates combined with a limited disinvestment phase in order to adjust in a smoother way the economy to the newly discovered reality of climate change.

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Appendix

A.1 Appendix 1

First consider a small positive variation $da > 0$ and $db > 0$ of $a(t)$ and $b(t)$ along the saddle branch SB_1. Then such variations should imply variations $da' > 0$ and $db' > 0$ of $a(t')$ and $b(t')$ located along the saddle branch for any time $t' > t$. Hence a simultaneous increase of $(\bar{a}, \bar{b})$ at $\bar{t}$ along SB_1 implies a simultaneous increase of $a(t)$ and $b(t)$ along the optimal trajectory at any $t > \bar{t}$. Remembering that $g^a(t) < 0$ and $g^b(t) < 0$ along the optimal trajectory corresponding to the saddle branch $SB1$:

$$\begin{aligned} a(t) &= a^* e^{-\int_t^\infty g^a(\tau)d\tau} = a^* e^{\int_t^\infty |g^a(\tau)|d\tau} \\ b(t) &= b^* e^{\int_t^\infty |g^b(\tau)|d\tau} \ . \end{aligned}$$

Differentiating and making use of the expressions (3.3) of g^a and (4.3) of g^b result in:

$$\begin{aligned} \frac{da(t)}{a(t)} &= \int_t^\infty d|g^a(\tau)|d\tau = -\int_t^\infty da(\tau)d\tau + \frac{\eta-\beta}{\eta}\int_t^\infty db(\tau)d\tau \\ \frac{db(t)}{b(t)} &= \int_t^\infty d|g^b(\tau)|d\tau = -\frac{1-\beta-\gamma}{1-\gamma}\int_t^\infty da(\tau)d\tau + (1-\beta)\int_t^\infty db(\tau)d\tau \ . \end{aligned}$$

Denote by:

$$Da(t) \equiv \int_t^\infty da(\tau)d\tau \quad \text{and} \quad Db(t) = \int_t^\infty db(\tau)d\tau \quad , \quad t \geq \bar{t} \ .$$

Solving the above linear system in $Da(t)$ and $Db(t)$, we obtain:

$$Da(t) = \frac{1}{\Delta}\left\{\frac{\eta-\beta}{\eta}\frac{db(t)}{b(t)} - (1-\beta)\frac{da(t)}{a(t)}\right\} \tag{A.1.1}$$

$$Db(t) = \frac{1}{\Delta}\left\{\frac{db(t)}{b(t)} - \frac{1-\beta-\gamma}{1-\gamma}\frac{da(t)}{a(t)}\right\} \ , \tag{A.1.2}$$

where $\Delta \equiv [\beta(1-\beta-\gamma+\eta\gamma)]/[\eta(1-\gamma)] > 0$. Since the point derivative at (a, b) of the optimal trajectories during the post ceiling phase is given by

bg^b/ag^a, we get $da(t)/a(t) = [|g^a(t)|]/[g^b(t)|(db(t)/b(t))$, thus:

$$\begin{aligned} Da(t) &= \frac{db(t)}{b(t)} \frac{1}{\Delta|g^b(t)|} \left\{ \frac{\eta-\beta}{\eta}|g^b(t)| - (1-\beta)|g^a(t)| \right\} \\ Db(t) &= \frac{db(t)}{b(t)} \frac{1}{\Delta|g^b(t)|} \left\{ |g^b(t)| - \frac{1-\beta-\gamma}{1-\gamma}|g^a(t)| \right\} . \end{aligned}$$

Making use of (3.3), (4.3) and the expressions (4.4) and (4.5) of a^* and b^*, straightforward manipulations show that:

$$\left\{ \frac{\eta-\beta}{\eta}|g^b(t)| - (1-\beta)|g^a(t)| \right\} = \Delta(a(t)-a^*) \implies Da(t) = \frac{db(t)}{b(t)} \frac{a(t)-a^*}{|g^b(t)|} \tag{A.1.3}$$

$$\left\{ |g^b(t)| - \frac{1-\beta-\gamma}{1-\gamma}|g^a(t)| \right\} = \Delta(b(t)-b^*) \implies Db(t) = \frac{db(t)}{b(t)} \frac{b(t)-b^*}{|g^b(t)|} \tag{A.1.4}$$

.

Since $a^* \leq a(t)$ and $b^* \leq b(t)$ along the saddle branch $SB1$ we conclude that $Da(t)$ and $Db(t)$ have the sign of $db(t)$. Since $a(t)$ and $b(t)$ vary in the same direction along the saddle branch, $Da(t)$ has the same sign as $da(t)$ and $Db(t)$ has the same sign as $db(t)$. As shown before this will imply that $da(t)$ and $da(t')$, $t \leq t'$ have the same sign and the same applies to $db(t)$ and $db(t')$. Hence $da(t)$ has the sign of $d\bar{a}$ and $db(t)$ has the sign of $d\bar{b}$, $t \geq \bar{t}$.

Next, the cumulated use of the resource over the post ceiling phase is given by:

$$\bar{X} = \int_{\bar{t}}^{\infty} x(t)dt = \bar{x} \int_{\bar{t}}^{\infty} e^{\int_{\bar{t}}^{t} g^x(\tau)d\tau} dt$$

Differentiating we obtain:

$$\frac{d\bar{X}}{\bar{X}} = \frac{d\bar{x}}{\bar{x}} + \frac{\int_{\bar{t}}^{\infty} \left[\int_{\bar{t}}^{t} dg^x(\tau)d\tau \right] e^{\int_{\bar{t}}^{t} g^x(\tau)d\tau} dt}{\int_{\bar{t}}^{\infty} e^{\int_{\bar{t}}^{t} g^x(\tau)d\tau} dt}$$

Multiplying and dividing the second term of the RHS by $\bar{x}$ results in:

$$\frac{d\bar{X}}{\bar{X}} = \frac{d\bar{x}}{\bar{x}} + \frac{\int_{\bar{t}}^{\infty} \bar{x} e^{\int_{\bar{t}}^{t} g^x(\tau)d\tau} \left[\int_{\bar{t}}^{t} dg^x(\tau)d\tau \right] dt}{\int_{\bar{t}}^{\infty} \bar{x} e^{\int_{\bar{t}}^{t} g^x(\tau)d\tau} dt}$$

Since $x(t) = \bar{x}e^{\int_{\bar{t}}^{t} g^x(\tau)d\tau}$ this is equivalent to:

$$\frac{d\bar{X}}{\bar{X}} = \frac{d\bar{x}}{\bar{x}} + \frac{\int_{\bar{t}}^{\infty} x(t)\left[\int_{\bar{t}}^{t} dg^x(\tau)d\tau\right]dt}{\int_{\bar{t}}^{\infty} x(t)}$$

Inverting the integration order and remembering that $\bar{X} = \int_{\bar{t}}^{\infty} x(t)dt$ and $X(t) = \int_{t}^{\infty} x(\tau)d\tau$:

$$\frac{d\bar{X}}{\bar{X}} = \frac{d\bar{x}}{\bar{x}} + \frac{\int_{\bar{t}}^{\infty} dg^x(t)\int_{t}^{\infty} x(\tau)d\tau dt}{\bar{X}} \tag{A.1.5}$$

$$= \frac{d\bar{x}}{\bar{x}} + \frac{\int_{\bar{t}}^{\infty} dg^x(t)X(t)dt}{\bar{X}} \tag{A.1.6}$$

We get from (4.1):

$$g^x(t) = \frac{1}{1-\gamma}[\delta - \beta a(t)] \implies dg^x(t) = -\frac{\beta}{1-\gamma}da \ .$$

Thus:

$$\frac{\partial g^x}{\partial \bar{K}} = -\frac{\beta}{1-\gamma}\frac{\partial a(t)}{\partial \bar{K}} = -\frac{\beta}{1-\gamma}\frac{\partial a(t)}{\partial \bar{b}}\frac{\partial \bar{b}}{\partial \bar{K}} > 0 \ , \tag{A.1.7}$$

and :

$$\frac{\partial g^x}{\partial \bar{x}} = -\frac{\beta}{1-\gamma}\frac{\partial a(t)}{\partial \bar{x}} = -\frac{\beta}{1-\gamma}\frac{\partial a(t)}{\partial \bar{b}}\frac{\partial \bar{b}}{\partial \bar{x}} < 0 \ . \tag{A.1.8}$$

Hence we obtain from (A.1.6):

$$\frac{\partial \bar{X}}{\partial \bar{K}} = \int_{\bar{t}}^{\infty} \frac{\partial g^x(t)}{\partial \bar{K}} X(t)dt > 0 \ ,$$

which proves the first part of the claim 1 of the Proposition P.1. Furthermore $X(t) \leq \bar{X}$ implies that:

$$\frac{\partial \bar{X}}{\partial \bar{x}} = \frac{\bar{X}}{\bar{x}} - \frac{\beta}{1-\gamma}\int_{\bar{t}}^{\infty} \frac{\partial a(t)}{\partial \bar{x}} X(t)dt \geq \frac{\bar{X}}{\bar{x}}\left\{1 - \frac{\beta\bar{x}}{1-\gamma}\frac{\partial \bar{b}}{\partial \bar{x}}\frac{\bar{a}-a^*}{\bar{b}|g^b(\bar{t})|}\right\} \ .$$

Since $\partial\bar{b}/\partial\bar{x} = \gamma\bar{b}/\bar{x}$, this is equivalent to:

$$\frac{\partial \bar{X}}{\partial \bar{x}} \geq \frac{\bar{X}}{\bar{x}}\left[1 - \frac{\beta\gamma}{1-\gamma}\frac{\bar{a}-a^*}{|g^b(\bar{t})|}\right]$$

$$= \frac{\bar{X}}{\bar{x}(1-\gamma)|g^b(\bar{t})|}\left[(1-\gamma)|g^b(\bar{t})| - \beta\gamma(\bar{a}-a^*)\right] \ .$$

Making use of (4.3), it is easily verified that the term into brackets is positive. Hence $\partial \bar{X}/\partial \bar{x} > 0$ proving the second part of the claim 1 of the Proposition P.1.

Next we get from (3.1), $\partial g^c(t)/\partial b(t) = \beta/\eta$ implying that $\partial g^c(t)/\partial \bar{K} = (\beta/\eta)(\partial b(t)/\partial \bar{b})(\partial \bar{b}/\partial \bar{K}) < 0$. Furthermore $\partial g^c(t)/\bar{x} = (\beta/\eta)(\partial b(t)/\partial \bar{b})(\partial \bar{b}/\partial \bar{x}) > 0$ which proves the claim 2 of the Proposition P.1.

Next, since $\bar{c} = \bar{a}\bar{K}$ by definition, we get $d\bar{c}/\bar{c} = d\bar{a}/\bar{a} + d\bar{K}/\bar{K}$. Thus:

$$\begin{aligned}\frac{\partial \bar{c}}{\partial \bar{K}} &= \frac{\bar{c}}{\bar{K}}\left[1 - (1-\beta)\frac{|g^a(\bar{t})|}{|g^b(\bar{t})|}\right] \\ &= \frac{\bar{c}}{\bar{K}|g^b(\bar{t})|}\left[|g^b(\bar{t})| - (1-\beta)|g^a(\bar{t})|\right] \ ,\end{aligned}$$

making use of $\partial \bar{a}/\partial \bar{K} = (d\bar{a}/d\bar{b})(\partial \bar{b}/\partial \bar{K}) = (|g^a(\bar{t})|/|g^b(\bar{t})|)(\beta - 1)$. It is easily verified that the expression into brackets is positive:

$$\begin{aligned} & |g^b(\bar{t})| > (1-\beta)|g^a(\bar{t})| \\ \Longleftrightarrow \quad & -\frac{1-\beta-\gamma}{1-\gamma}\bar{a} + (1-\beta)\bar{b} - \frac{\delta}{1-\gamma} > (1-\beta)\left[-\bar{a} + \frac{\eta-\beta}{\eta}\bar{b} + \frac{\rho}{\eta}\right] \\ \Longleftrightarrow \quad & -(1-\beta-\gamma)\bar{a} + (1-\beta)(1-\gamma)\bar{b} - \delta > -(1-\beta)(1-\gamma)\bar{a} \\ & \qquad\qquad\qquad + (1-\beta)(1-\gamma)\left[(1-\frac{\beta}{\eta})\bar{b} + \frac{\rho}{\eta}\right] \\ \Longleftrightarrow \quad & -\delta > -\beta\gamma\bar{a} - (1-\beta)(1-\gamma)\frac{\beta}{\eta}\bar{b} + (1-\beta)(1-\gamma)\frac{\rho}{\eta} \\ \Longleftrightarrow \quad & \delta\eta < \gamma\beta\eta\bar{a} + \beta(1-\beta)(1-\gamma)\bar{b} - (1-\beta)(1-\gamma)\rho \\ \Longleftrightarrow \quad & \delta\eta + (1-\beta)(1-\gamma)\rho < \gamma\beta\eta\bar{a} + \beta(1-\beta)(1-\gamma)\bar{b}\end{aligned}$$

Since $\gamma\beta\eta a^* + \beta(1-\beta)(1-\gamma)b^* = \delta\eta + (1-\beta)(1-\gamma)\rho$ making use of the expressions (4.4) and (4.5) of a^* and b^* and $a^* < \bar{a}$, $b^* < \bar{b}$, we conclude that the inequality is verified. Thus $\partial \bar{c}/\partial \bar{K} > 0$. Next $\partial \bar{c}/\partial \bar{x} = (\partial \bar{a}/\partial \bar{b})(\partial \bar{b}/\partial \bar{x}) > 0$ implies that $\partial \bar{c}/\partial \bar{x} > 0$ and proves the claim 3 of the Proposition P.1.

Next since $dg^x(t) = -(\beta/(1-\gamma))da(t)$, $\partial a(t)/\partial \bar{K} < 0$ and $\partial a(t)/\partial \bar{x} > 0$ imply together that $\partial g^x(t)/\partial \bar{K} > 0$ and $\partial g^x(t)/\partial \bar{x} < 0$, that is the claim 4 of the Proposition P.1. Last $dg^K = db - da = db(1 - da/db)$. Since db/da the slope of the saddle branch is higher than the slope of the locus $\dot{a} = 0$, $db/da > \eta/(\eta-\beta) > 1$. Thus $da/db < 1$ implies that dg^K has the same sign as $db(t)$, that is the sign of $d\bar{b}$. Hence $\partial g^K(t)/\partial \bar{K} < 0$ and $\partial g^K(t)/\partial \bar{x} > 0$, that is the claim 5 of the proposition. The proof of the Proposition P.1. is now complete. ■

A.2 Appendix 2

Consider first the effects of a higher initial resource endowment, $d\underline{X} > 0$. Since $\underline{b}$ is unchanged, the geometry of the phase plane defines the same increasing relationship between $\bar{t}$ and $\underline{a}$, that is $\bar{t}_0(\underline{a})$ is not modified by an increase of $\underline{X}$. Differentiating the stock condition:

$$\left[\bar{x} + \bar{X}'(\bar{K})\frac{\delta \bar{K}}{1-\beta}\right] d\bar{t} \;=\; d\underline{X} + \bar{X}'(\bar{K})\frac{\bar{K}}{1-\beta}\frac{\partial \bar{b}/\partial \underline{a}}{\bar{b}} d\underline{a}$$

shows that the curve $\bar{t}_1(\underline{a})$ is shifted upward by a higher $\underline{X}$. Hence, both $\underline{a}$ and $\bar{t}$ are shifted upward by a higher $\underline{X}$. To a higher $\underline{a}$ corresponds a higher initial consumption rate $\underline{c}$ and a lower $b(t)$ over the time interval $[\underline{t}, \bar{t})$, that is a lower consumption growth rate, since $g^c(t) = (\beta b(t) - \rho)/\eta$, showing the first claim of the Proposition P. 2.

Turn to the effect of a higher initial capital stock, $d\underline{K} > 0$. The differentiation of the stock condition gives:

$$\left[\bar{x} + \bar{X}'(\bar{K})\frac{\delta \bar{K}}{1-\beta}\right] d\bar{t} \;=\; -\bar{X}'(\bar{K})\frac{\bar{K}}{\underline{K}}\frac{\underline{b}}{\bar{b}}\frac{\partial \bar{b}}{\partial \underline{b}} d\underline{K} + \bar{X}'(\bar{K})\frac{\bar{K}}{1-\beta}\frac{\partial \bar{b}/\partial \underline{a}}{\bar{b}} d\underline{a}\ .$$

Since $\partial \bar{b}/\partial \underline{b} > 0$, the curve $\bar{t}_1(\underline{a})$ is shifted down by a higher capital endowment $\underline{K}$. On the other hand, a larger $\underline{K}$ is equivalent to a lower $\underline{b}$. To any given $\underline{a}$, the trajectories in the phase plane are thus shifted down, corresponding to a slower move and thus a higher $\bar{t}$. This shows that the curve $\bar{t}_0(\underline{a})$ is shifted up by a higher $\underline{K}$. The consequences of these moves is a decrease of $\underline{a}^*$ but the effect over $\bar{t}^*$ seems at first sight indeterminate.

Fix $\bar{t}$, denote by $d\underline{a}_1$ the horizontal variation of $\underline{a}$ induced by the shift of the $\bar{t}_1(\underline{a})$ function and similarly denote by $d\underline{a}_0$ the horizontal variation of $\underline{a}$ induced by the shift of $\bar{t}_0(\underline{a})$. As shown before, $d\underline{a}_0 < 0$ and $d\underline{a}_1 < 0$. Hence, if $|d\underline{a}_0| < |d\underline{a}_1|$ then the shifted value of $\bar{t}$ at the intersection between the shifted curves $\bar{t}_0(\underline{a})$ and $\bar{t}_1(\underline{a})$ should be located below the original $\bar{t}$. Hence, $|d\underline{a}_0| < |d\underline{a}_1| \iff d\bar{t} < 0$. To prove that this is effectively the case, remark that for a fixed $\bar{t}$ level, the stock constraint implies that $\bar{K}$ should remain constant and thus $\bar{b}$, that is: $d\bar{b} = \partial\bar{b}/\partial\underline{a}d\underline{a}_1 + \partial\bar{b}/\underline{b}d\underline{b} = 0$. On the other hand, a fixed $\bar{t}$ requires to shift the original trajectory in the phase plane when $\underline{b}$ is increased to a position located below the original curve. In the contrary case, the move toward the SB_1 manifold would be accelerated. But this implies that $\bar{b}$ should decrease and thus that: $d\bar{b} = \partial\bar{b}/\partial\underline{a}d\underline{a}_0 + \partial\bar{b}/\partial\underline{b}d\underline{b} < 0$. Since $\partial\bar{b}/\partial\underline{a} < 0$, it is immediate that $|d\underline{a}_0| < |d\underline{a}_1|$ and thus $d\bar{t}^*/d\underline{K} < 0$.

Differentiating $\underline{c} = \underline{aK}$ results in:

$$\begin{aligned}\frac{d\underline{c}}{\underline{c}} &= \frac{d\underline{K}}{\underline{K}}\left[1-(1-\beta)\frac{d\underline{a}}{d\underline{b}}\frac{\underline{b}}{\underline{a}}\right]\\ &= \frac{d\underline{K}}{\underline{K}}\left[1-(1-\beta)\frac{g^a(\underline{a},\underline{b})}{g^b(\underline{a},\underline{b})}\right]\end{aligned}$$

Making use of the expressions (3.3) of g^a and (5.1) of g^b during the ceiling phase, it is immediately checked that:

$$\frac{d\underline{c}}{d\underline{K}} > 0 \iff \frac{g^b(\underline{a},\underline{b})}{g^a(\underline{a},\underline{b})} > 1-\beta \iff \underline{b} > \hat{b}$$

Since $\underline{b} > \hat{b}$ for the type I family of trajectories, we conclude that $d\underline{c}/d\underline{K} > 0$. Next since $\underline{a}$ and $\underline{b}$ are shifted downward by an upward shift of $\underline{K}$, they generate a shifted trajectory in the phase plane located below the original one, that is $b(t)$ is shifted downward when $\underline{K}$ is increased. Since $g^c(t) = (\beta b(t) - \rho)/\eta$, we conclude that $dg^c(t)/d\underline{K} < 0$, completing the proof of the claim 2 of the Proposition P. 2.

Turning to the effects of an increase of $\bar{x}$, the stock condition implies that:

$$\left[\bar{x} + \frac{\delta\bar{K}}{1-\beta}\right]d\bar{t} + \left[\bar{t} + \frac{\partial\bar{X}}{\partial\bar{x}}\right]d\bar{x} \quad = \quad \frac{\partial\bar{X}}{\partial\bar{K}}\frac{\bar{K}}{1-\beta}\frac{\partial\bar{b}}{\partial\underline{a}}d\underline{a}$$

This shows that the curve $\bar{t}_1(\underline{a})$ is shifted down by a higher $\bar{x}$. On the other hand, to a higher $\bar{x}$ corresponds a higher $\underline{b}$. Hence for any given $\underline{a}$, the connecting trajectory in the phase plane is translated upward and $\bar{a}$ is reduced. This results in an accelerated move along a shorter trajectory, that is $\bar{t}$ is decreased. Hence a higher $\bar{x}$ shifts down the curve $\bar{t}_0(\underline{a})$. The two curves $\bar{t}_0(\underline{a})$ and $\bar{t}_1(\underline{a})$ being both shifted down, it is immediate that $\bar{t}^*$ is decreased, that is $d\bar{t}^*/d\bar{x} < 0$. But the effect over $\underline{a}$ appears indeterminate and thus no definite conclusion arises concerning the qualitative effect of an upper shift of $\bar{x}$ on $\underline{c} = \underline{aK}$. The same applies to the effect of a shift of $\bar{x}$ upon $b(t)$ and hence upon $g^c(t)$. This proves the claim 3 and complete the proof of the Proposition P.2. $\blacksquare$